THE TWENTY-SECOND ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-second annual meeting of the Society was held in New York City on Monday and Tuesday, December 27–28, 1915. The attendance at the four sessions included the following seventy-two members:

Mr. J. W. Alexander II, Professor Clara L. Bacon, Dr. Ida Barney, Professor R. D. Beetle, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor F. N. Cole, Professor J. L. Coolidge, Professor D. R. Curtiss, Professor L. E. Dickson, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Dr. C. A. Fischer, Professor W. B. Fite, Professor Tomlinson Fort, Dr. Meyer Gaba, Mr. W. Van N. Garrettson, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. Edward Kircher, Professor J. K. Lamond, Dr. D. D. Leib, Dr. P. H. Linehan, Dr. Joseph Lipka, Professor W. R. Longley, Professor C. R. MacInnes, Dr. W. E. Milne, Professor H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Dr. G. A. Pfeiffer, Professor H. B. Phillips, Professor Arthur Ranum, Professor L. H. Rice, Professor R. G. D. Richardson, Dr. P. R. Rider, Mr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor Clara E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Elijah Swift, Mr. H. S. Vandiver, Professor E. E. Whitford, Dr. C. E. Wilder, Professor Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professor Edward Kasner. The Council announced the election of the following persons to membership in the Society: Professor W. E. Edington, University of New Mexico; Professor J. L. Gibson, University

of Utah; Dr. W. E. Milne, Bowdoin College; Professor L. J. Reed, University of Maine. Nine applications for membership in the Society were received.

The total membership of the Society is now 732, including 73 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 418; the number of papers read was 197. The number of members attending at least one meeting during the year was 253. At the annual election 204 votes were cast. The Treasurer's report shows a balance of $10,470.58, including the life membership fund of $5,560.30. Sales of the Society's publications during the year amounted to $1,832.93. The Library now contains about 5,250 volumes, excluding unbound dissertations.

The Society received with great regret the resignation of Professor L. E. Dickson from the Editorial Committee of the *Transactions*, to take effect October 1, 1916, at the end of fifteen years of editorial services including six years as member of the Editorial Committee. Committees were appointed by the Council to nominate successors to Professor Dickson and Professor D. R. Curtiss, whose first term as member of the Editorial Committee expires on October 1, 1916.

Sixty members and friends attended the annual dinner of the Society on Monday evening.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor E. R. HEDRICK, Professor VIRGIL SNYDER.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1918,

Professor G. A. BLISS,
Professor R. D. CARMICHAEL,
Professor W. B. FITE,
Professor F. S. WOODS.

The following papers were read at this meeting:

(1) Mr. J. F. RITT: "On the derivatives of a function at a point."

(2) Mr. J. F. RITT: "The finite groups of a class of functions of a real variable."

(3) Professor J. E. ROWE: "A new method of deriving the equation of a rational plane curve from its parametric equations."

(4) Professor H. B. PHILLIPS: "Elastic nets."

(5) Professor ARTHUR RANUM: "The singular points of analytic space curves."

(6) Dr. H. M. SHEFFER: "The reduction of non-monadic relations to monadic" (preliminary communication).

(7) Dr. H. M. SHEFFER: "The elimination of modular existence postulates."

(8) Professor BESSIE I. MILLER: "A new canonical form of the elliptic integral."

(9) Mr. A. R. SCHWEITZER: "On the use of supernumerary indefinables in the construction of axioms."

(10) Professor TOMLINSON FORT: "Linear difference and differential equations."

(11) Dr. DUNHAM JACKSON: "Algebraic properties of self-adjoint systems."

(12) Professor G. D. BIRKHOFF: "On dynamical systems with two degrees of freedom."

(13) Professor G. D. BIRKHOFF: "Infinite products of analytic matrices."

(14) Professor E. B. WILSON: "Ricci's absolute calculus and its application to the theory of surfaces."

(15) Professor C. L. E. MOORE: "Some theorems regarding two-dimensional surfaces in euclidean n-space."

(16) Dr. OLIVE C. HAZLETT: "On the fundamental invariants of nilpotent algebras in a small number of units."

(17) Dr. EDWARD KIRCHER: "Some properties of finite algebras."

(18) Professor M. FRÉCHET: "On Pierpont's definition of integrals."

(19) Professor EDWARD KASNER: "Infinite groups of conformal transformations."

(20) Dr. JOSEPH LIPKA: "Isogonal, natural, and isothermal families of curves on a surface."

(21) Dr. L. L. SILVERMAN: "On the consistency and

equivalence of certain generalized definitions of the limit of a function of a continuous variable."

(22) Professor L. P. EISENHART: "Ruled surfaces generated by the motion of an invariable curve."

(23) Professor L. P. EISENHART: "Transformations of surfaces Ω (second paper)."

(24) Dr. G. M. GREEN: "On rectilinear congruences and nets of curves on a surface."

(25) Professor W. F. OSGOOD: "On infinite regions."

(26) Professor JOHN EIESLAND: "On sphere-flat geometry."

(27) Professor J. L. COOLIDGE: "The meaning of Plücker's numbers for a real curve."

(28) Professor W. M. SMITH: "Characterization of the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force."

(29) Professor H. H. MITCHELL: "On the generalized Jacobi-Kummer cyclotomic function."

(30) Professor H. H. MITCHELL: "On the congruence $cx^\lambda + 1 \equiv dy^\lambda$ in a Galois field."

(31) Professor R. D. BEETLE: "Sets of properties characteristic of the arithmetic and geometric means."

(32) Dr. R. L. MOORE: "On the foundations of geometry."

(33) Dr. W. C. GRAUSTEIN: "The correspondence of space curves by the transformation of Combescure and by a transformation thereby suggested."

(34) Mr. R. E. GLEASON: "On Dirichlet's principle."

(35) Dr. W. E. MILNE: "On the degree of convergence of Birkhoff's series."

(36) Professor G. C. EVANS: "A generalization of Bôcher's analysis of harmonic functions" (preliminary report).

(37) J. W. ALEXANDER, II: "On the factorization of plane Cremona transformations."

(38) Mr. L. B. ROBINSON: "On elimination between several polynomials in several variables."

Professor Fréchet's paper was communicated to the Society through Professor Curtiss; Mr. Gleason was introduced by Professor Birkhoff. In the absence of the authors Professor Wilson's paper was read by Professor C. L. E. Moore, and the papers of Dr. Sheffer, Professor Miller, Mr. Schweitzer, Professor Fréchet, Dr. Silverman, Professor Coolidge, Dr. Graustein, Professor Evans, Mr. Alexander, and Mr. Robinson were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Ritt showed in an earlier paper that, if x is confined to the real domain, the function

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{n!} \left(1 - e^{-\frac{1}{b_n x^2}}\right) \qquad (1 < b_n > |a_n|)$$

has a_n as its nth derivative for $x = 0$, even if

$$\text{lim. sup. } \sqrt[n]{|a_n|/n!} = \infty.$$

This furnished the solution of a problem suggested by Professor Kasner, for which other solutions had been given by Borel and by Serge Bernstein.

In the present paper the domain of the variable is extended over a sector of the complex domain, and a solution of a problem proposed several years ago by Van Vleck is thereby obtained.

By a slight modification of the function above it is shown that a series of negative powers alone can be found which has a_n as its nth derivative for $x = 0$ relative to a sector of the plane.

An extension to the case of several variables is made.

2. In a former paper "On Babbage's functional equation," Mr. Ritt considered the cyclic groups of a class of functions in which the real linear fractional functions are included.

In the present paper it is shown that the only groups of functions of this class are the cyclic and the dihedral. It is shown also that any two isomorphic groups can be transformed into each other. The distribution of certain critical points connected with the group is studied. Applications are made to the theory of real linear transformations.

3. The neatest form of the equation of a rational plane curve of degree n, which we call the R^n, that may be derived from its parametric equations is found by equating to zero the n-rowed determinant obtained by equating to zero the Bezout eliminant of two line sections of the R^n, together with the application of a well-known translation scheme. The purpose of Professor Rowe's paper is to show how this same

equation may be derived from the equation of the line determined by two points of the R^n. This method involves neither the Bezout eliminant nor the translation scheme.

4. Take a finite number of points in a space of any number of dimensions. Let certain pairs of the points attract each other with a force proportional to the distance. The factors of proportionality for different pairs need not be the same. Let some of the points, called fixed, be held in arbitrarily assigned positions, while the others, called free, adjust themselves in positions of equilibrium. If each pair of points is connected by at least one chain of free points, each attracting the preceding and following, Professor Phillips calls the resulting configuration an elastic net. There is a unique position of equilibrium. The net has some curious properties. For example, if any free point is displaced from its equilibrium position, the force tending to return it is proportional to the displacement and independent of the positions of the fixed points. If any point, free or fixed, is moved, none of the free points can remain at rest, but all move in the same direction distances independent of the positions of the fixed points.

5. In 1901 Burali-Forti classified the singular points of analytic space curves not only with respect to the evenness or oddness of the exponents λ, μ, ν in the expansions

$$x = au^{\lambda} + \cdots, \qquad y = bu^{\mu} + \cdots, \qquad z = cu^{\nu} + \cdots,$$

but also with respect to the values of the curvature $1/r$ and the torsion $1/\rho$ of the curves at the singular points; and by combining the two principles he found fifty classes of such points. His classification, however, has the disadvantage of not being self-dual. In the present note Professor Ranum remedies this defect by taking into account the plane curvature $1/r'$ (defined and discussed by him in a recent number of the *Quarterly Journal*), which is the dual of the point curvature $1/r$. One result is that the number of classes is increased to seventy-four.

6. A postulate set for a deductive system is usually based on one or more undefined classes and one or more undefined relations, each relation being at least dyadic (operations are a special type of relation). By means of the notion of (1, 2)

correspondence Dr. Sheffer shows (1) how to reduce non-monadic relations to monadic; and, thus, (2) how to base postulate sets exclusively on (1, 2) correspondences and classes.

7. Postulate sets for groups, fields, Boolean algebras, and number algebras usually contain one or more existence postulates of the form

(α) There is a K-element x such that $x \circ x = x$,

or of the form

(β) There is a K-element x such that, for every K-element a,

$$a \circ x = x \circ a = a.$$

(Examples: $x =$ the addition modulus, 0; $x =$ the multiplication modulus, 1.) Postulates of type (α) or of type (β) may be called modular existence postulates. Dr. Sheffer shows how, in the construction of postulate sets, modular existence postulates may always be avoided.

8. A particular curve of the family of elliptic norm curves Q_n in S_{n-1} which admit a group G_{2n^2} of collineations is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod n).

In the group G_{2n^2} there are certain involutorial collineations with two fixed spaces. If Q_n is projected from one upon the other, Q_n is mapped by a family of rational curves R_m with the parameter t. Under certain conditions the quadratic irrationality separating involutorial points on Q_n can define the elliptic parameter

$$u = \int \frac{(tdt)}{\sqrt{(t\tau)\alpha_\tau{}^{n-3}\alpha_t{}^3}}.$$

In Professor Miller's paper the form u is studied, is contrasted with Klein's form, and its natural occurence in connection with Q_3, Q_4, Q_5 is discussed.

9. In his well-known set of axioms for euclidean geometry Hilbert uses more undefined relations than are necessary for the construction of the desired geometric properties, i. e., on the basis of his system certain of his primitive relations may be defined in terms of others. The possible logical advantage

of such supernumerary indefinables, however, Hilbert does not bring clearly to view. The purpose of Mr. Schweitzer's note is to point out that supernumerary indefinables may sometimes be employed to eliminate explicit reference to existential properties in axioms; this is illustrated by his set of axioms for a field.*

10. Professor Fort's paper is divided into three parts. In the first part difference and differential equations are set up whose coefficients obey certain laws of which periodicity is a very special case and some fundamental theorems are proved for equations of this character.

Part two extends some familiar oscillation theorems primarily due to Sturm to equations of the type considered and proves the continuity of the characteristic values λ_j, for fixed boundary conditions and an interval of fixed length but of variable position, the characteristic values being considered as functions of the initial point of the interval.

Part three considers the so-called self-adjoint boundary value problems for the differential and difference equations where both coefficients depend upon a parameter λ. It is proved that the extremes of a certain type of functions λ_j considered under part two are the characteristic values for the problem in hand. The existence and number of these characteristic values is discussed, and means for distinguishing various cases, etc., taken up. A theorem of oscillation is given.

11. The general definition of adjoint boundary conditions associated with ordinary linear differential equations was given by Birkhoff (*Transactions*, 1908). Bôcher (*Transactions*, 1913) has investigated the circumstances under which a system of the second order is self-adjoint. In Dr. Jackson's paper a condition is given for self-adjoint sets of boundary conditions associated with differential equations of any order. The condition is expressed by a matrix equation of simple form, involving the given coefficients. It becomes particularly symmetric if the given differential equation itself is self-adjoint or anti-self-adjoint.

12. Professor Birkhoff begins by reducing the equations of motion for a dynamical system with two degrees of freedom

* Cf. BULLETIN, vol. 21 (1914–1915), p. 295.

to a special form which he has earlier employed.* In this way he is able to develop simple and general criteria for the existence of periodic orbits of a certain *primary* type. With the aid of these orbits a reduction of the dynamical problem to a problem in the transformation of surfaces into themselves is effected in a large class of cases. This reduction affords a means of proving the existence of infinitely many *secondary* periodic orbits, and of determining the precise structure of certain recurrent orbits and of orbits asymptotic to periodic and recurrent orbits. Finally the structure of the general orbit is determined. A number of applications of the theorems to special dynamical problems are given.

13. In a large part of the theory of functions of a single complex variable, the matrix of analytic functions rather than the single analytic function must be considered as the fundamental element. This is certainly the case for the functions defined by linear differential and difference equations.

The goal of the second paper by Professor Birkhoff is to show that the classical theorems of Weierstrass and Mittag-Leffler treating of the formation of functions with singularities of assigned type admit of a natural extension to matrices of analytic functions.

14. Professor Wilson calls attention to Ricci's generally neglected absolute calculus and to its suggestiveness as an implement of research in developing the theory of surfaces of two dimensions in euclidean space of n dimensions.

15. Professor Moore shows that the mean curvature of a two-dimensional surface in higher dimensions than three must be regarded as a vector magnitude and that the properties connected with the curvature of the surface may be expressed very simply with reference to an ellipse in the normal space, the vector from the surface point to the center of the ellipse being the mean curvature.

16. One of the two main outstanding problems in the theory of linear algebras is that of the invariantive classification of nilpotent algebras—that is, algebras such that some power of every number in the algebra is zero. In this paper Dr.

* *Rendiconti del Circolo Matematico di Palermo*, vol. 39, pp. 265–334.

Hazlett considers rational integral invariants of such algebras. All rational integral invariants for n-ary linear algebras under the total linear group reduce to zero for a nilpotent algebra, and accordingly we consider invariants under the group which leaves unaltered the canonical form. For such invariants this paper proves theorems analogous to theorems about invariants of algebraic forms, and in particular proves the finiteness of the rational integral invariants. The fundamental invariants are found for the simpler cases.

17. Vandiver and Fraenkel have studied finite algebras, but the former did not take up the question of factorization in such an algebra, while Fraenkel in the main restricts himself to decomposable (zerlegbar) rings or algebras. In this paper Dr. Kircher takes up the case of any finite algebra whose elements combine by addition and multiplication subject to the commutative, associative, and distributive laws, the algebra possessing unit elements with regard to both addition and multiplication. Division is not always possible, and when possible is not necessarily unique. Since every algebra fulfilling these conditions can always be represented by the residue classes of a modular system the subject is developed from this point of view. Since the law of unique factorization fails, a partial restoration is effected by the introduction of ideals among which a certain type defined as absolute prime ideals is of great use. These have the same relation to prime ideals in the algebra that an absolute prime modular system has to the irreducible modular system containing it. A number of theorems analogous to well-known number theory theorems are also obtained.

18. Professor Fréchet's paper appears in full in the present issue of the BULLETIN.

19. The groups discussed by Professor Kasner are related to the two types of conformal transformation of order two presented in a paper read at the Providence meeting and there called conformal symmetries and conformal involutions. The first type reverses the orientation of angles while the second preserves it. The groups are infinite, in the sense of involving an infinite number of parameters, and contain subgroups generated by operators of period two.

20. The principal theorem proved in Dr. Lipka's paper is a generalization of the geometric characterization of isogonal trajectories on a surface given by the author in the *Annals of Mathematics*, volume 15, No. 2. The generalized theorem states that if from the geodesic curvature of the ∞^2 geodesic curvature elements composing an isogonal family we subtract the geodesic curvature of the corresponding ∞^2 elements of *any* other isogonal family, and then rotate each element through a right angle, the ∞^2 new elements will form a natural family. This gives a general test for an isogonal family of curves on a surface.

21. In this paper Dr. Silverman establishes a correspondence between certain functions $f(z)$ and certain generalized definitions of the limit of a function of a continuous variable. The results obtained are similar to those presented to the Society in September, 1914, by Silverman and Hurwitz, who established a correspondence between certain functions $f(z)$ and certain definitions of summability of a divergent sequence. As in the case of sequences the following propositions are proved: (1) if $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, the corresponding definition is regular, i. e., correctly evaluates any existing limit of a function of a continuous variable; (2) all such definitions are consistent, i. e., if two of them furnish generalized limits of the same function, the values are the same; (3) two definitions are equivalent, i. e., have exactly the same scope of application, provided the corresponding functions have the same zeros with the same multiplicities in C; (4) the definitions for the limit of a function of a continuous variable, corresponding to those of Cesàro and Hölder of the same order for sequences, are equivalent. The last result is not new, having been first proved by Landau in 1913; but it appears here as a special case of (3).

22. One of the outstanding problems in the theory of surfaces is the determination of those surfaces which may be generated in two ways by the motion of invariable curves. The quadrics and surfaces of translation are well-known examples of such surfaces. Professor Eisenhart proposes and solves the problem of finding all ruled surfaces generated by the motion of an invariable curve whose points describe the

generators in the motion. It is found that cylinders and right conoids are the only surfaces possessing this property. A right conoid is a surface whose generators meet an axis to which they are perpendicular. A circular cylinder with the axis of the conoid for an element meets the conoid in a curve which goes into a congruent curve on the surface as the cylinder rolls on the envelope of the family of circular cylinders with the same radii, each having the axis of the conoid for an element.

23. Professor Eisenhart's second paper appeared in full in the January number of the *Transactions*.

24. Starting with any non-conjugate net N on a curved surface S, and a line g passing through each point of the surface and not lying in the tangent plane of that point, Dr. Green associates with the congruence Γ of lines g a second congruence Γ' bearing a certain characteristic relation R to the congruence Γ. To every point P of S and the line g through it, corresponds a line g' of the congruence Γ' lying in the tangent plane to S at P, and not passing through P. The relation R between the congruences Γ and Γ' is uniquely determined by the net of parameter curves N; if N is altered, the congruence Γ' is in general also changed. If N is a conjugate net, the relation R subsists between what Wilczynski* has called the axis and ray congruences. If N is not conjugate, the generalized axis and ray congruences as defined by Dr. Green in another paper† are also in the relation R. Probably the most remarkable case in which the relation R exists is that in which the net N is the asymptotic net, and the congruences Γ and Γ' are Wilczynski's directrix congruences. In this connection is given a new geometric characterization of these congruences: if N is asymptotic, then two congruences Γ and Γ' in the relation R have their developables in correspondence if and only if Γ and Γ' are the directrix congruences. Applications of the preceding ideas are made to the general theory of congruences and to the theory of surfaces.

25. Professor Osgood's paper gives a general definition of infinite regions by which ordinary complex n-dimensional space may be closed, projective space, the space of the geometry of

* *Transactions*, vol. 16 (July, 1915), pp. 311–317.
† Cf. the abstract in the July BULLETIN, vol. 21 (1914–15), pp. 484–5.

inversion, and the space of analysis being the most familiar examples. All such extended spaces are linearly simply connected. An extension of Weierstrass's theorem is then obtained, whereby a function which is meromorphic at every point of such a closed space is rational. It follows that any transformation of such a space into itself, which is regular at every point, is birational, and the Jacobian cannot vanish.

26. Professor Eiesland's paper is a continuation of the author's memoir in the *American Journal of Mathematics*, volume 35, entitled, "On a flat spread-sphere geometry in odd-dimensional space."

In the first part of the paper a method of obtaining surfaces with coordinate lines of curvature is given, based on two theorems which are proved. Surfaces whose lines of curvature are plane in all $n - 2$ systems have been derived and particularly the molding surfaces in S_{n-1} and the generalized Dupin cyclides of the third and fourth order.

The second part of the paper deals with the sphere-flat transformations, and surfaces with coordinate asymptotic lines.

The asymptotic lines on a sphere in $\bar{S}_{n-1}$ are obtained by the integration of a system of differential equations. This integration problem reduces to that of a Riccati equation and quadratures.

27. The usual geometric definition for the Plücker numbers of a plane curve, order, class, etc., is such as to require the recognition of both real and imaginary elements. In Professor Coolidge's paper new definitions are found which are suitable to the case where the universe of discourse includes only the real elements of the plane.

28. Professor Smith's paper shows that the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force are completely characterized by two properties: (1) That the trajectories be isothermal; (2) That the isoclines be straight lines. By isoclines is meant the curves formed by joining points on consecutive curves which have parallel tangents.

29. If α is a λth root of unity, where λ is any integer, and

q a prime of the form $\lambda\nu + 1$, the Jacobi function $\psi(\alpha)$ has the property that $\psi(\alpha)\psi(\alpha^{-1}) = q$. Professor Mitchell discusses a more general function such that $\psi(\alpha)\psi(\alpha^{-1}) = q^t$, where q is any prime not contained in λ, and t any exponent for which $q^t \equiv 1$, mod. λ. For the case where λ is prime and t is the exponent to which q belongs, mod. λ, this function has been considered by Kummer. The present author determines the ideal factors of the function by essentially the same method which Kummer used in the special case mentioned. An error of Kummer's in this determination is found. Certain properties of these functions are also established.

30. The function $\psi(\alpha)$ considered in Professor Mitchell's first paper bears the same relation to the congruence $cx^\lambda + 1 \equiv dy^\lambda$ for the Galois field of order q^t that the Jacobi function does for the set of integral residues, mod. q, where q is a prime of the form $\lambda\nu + 1$. It is shown in his second paper that the number of solutions of any such congruence is determined if the functions $\psi(\alpha)$ are known. Two applications of this result are made, one to obtain an expression for the number of solutions of such a congruence in a field of order q^{st} in terms of those for a field of order q^t, and another to determine the exact values of these numbers for a field of order q^{2st}, provided $q^t \equiv -1$, mod. λ.

31. If $x_1, x_2, \cdots, x_n$ are n observed values, presumably equally accurate, of a magnitude which has an exact, but unknown, value, it is customary to regard the arithmetic mean of the observed values as the most probable value of the magnitude determined by them. A number of writers have attempted to justify this practice by selecting a set of properties which the most probable value ought to possess, and then proving that the set of properties characterizes the arithmetic mean. In his paper, Professor Beetle presents a set of three properties which characterizes the arithmetic mean, and also a set of three properties which characterizes the geometric mean.

If we denote the most probable value determined by $x_1, x_2, \cdots, x_n$ by $f_n(x_1, x_2, \cdots, x_n)$, the three properties which characterize the arithmetic mean are

(1) $f_n(x, x, x, \cdots, x) = x$;

(2) $f_n(x_1, x_2, \cdots, x_n)$ is a symmetric function of its arguments;

(3) $f_n(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

$$= f_n(x_1, x_2, \cdots, x_n) + f_n(y_1, y_2, \cdots, y_n).$$

The first two of the three properties characteristic of the geometric mean are the same as the first two for the arithmetic mean. The third is

(3′) $f_n(x_1y_1, x_2y_2, \cdots, x_ny_n)$

$$= f_n(x_1, x_2, \cdots, x_n)f_n(y_1, y_2, \cdots, y_n).$$

These sets of properties are not only extremely simple, but also exhibit very clearly the essential difference between the arithmetic mean and the geometric mean. In each set, the three properties are completely independent.

32. Concerning Hilbert's group-theoretic treatment of the foundations of geometry,* Poincaré says (according to Halsted's translation†): "Without doubt this is still not entirely satisfactory since though the *form* of the group is supposed any whatever, its *matter*, that is to say, the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, ..."

The treatment to which Poincaré here refers is based on three axioms (Axioms I, II and III). In an abstract‡ of a paper presented to the Society in April, Dr. Moore proposed a set of 12 axioms (Axioms 1–12) for plane analysis situs. These axioms are in terms of *point* and *region*. He desires to show that if to Axioms 1, 2, 4–7, 9–11 there be added the following Axioms *A* and *B* in addition to Hilbert's Axioms I, II and III (interpreted so as to apply properly in this new setting), then every space that satisfies the thus obtained set of axioms is either a euclidean or a Bolyai-Lobachevskian space of two-dimensions according as the group of all motions does or does not contain an invariant subgroup. This treatment (based on a set of axioms involving the notions point, region, and motion) does not presuppose that space, or even that any part of space, is a number manifold.

* D. Hilbert, "Ueber die Grundlagen der Geometrie," *Math. Annalen*, vol. 56 (1902).

† Cf. *Science*, May 19, 1911, p. 765.

‡ Cf. this BULLETIN, vol. 21 (July, 1915), pp. 485–486.

A. If O is a point of a region R then there exists a region K containing O and such that K plus its boundary is a subset of R.

B. If A and B are distinct points and every region containing the point O contains a point of the point set M then there exists a point P belonging to M and such that every region containing A and B can be moved into a point set containing O and P.

33. Two curves corresponding point for point so that the tangents at corresponding points are parallel are said to be related by a transformation of Combescure. This transformation finds an analytic parallel in a second point-to-point correspondence of two curves, in which the tangents at corresponding points have as their common perpendicular direction that of the principal normal at the point of the given curve and make with one another an angle whose cosine is equal to the ratio of the corresponding elements of arc, so that the element of arc of the given curve is the projection of that of the transformed curve. The general and special correspondences of these two types are discussed in Dr. Graustein's paper.

34. The object of Mr. Gleason's paper is to establish Dirichlet's principle in n dimensions ($n \geq 2$) under very general conditions by a new method. The method consists partly in the formation of a function Ψ from a function V, approximately satisfying the given boundary values, by assigning to Ψ, as its value at any point of its region of definition, the weighted mean of V throughout an n-dimensional sphere with center at this point—the sphere being divided into concentric weighted spherical shells.

The n-dimensional region considered is bounded. The (inner) surface distribution v, on its boundary S is required to be limited and continuous except possibly at a set of points (π) whose projected "area" on the coordinate hyperplanes is of outer content zero. Then (1) if the number of times a point P of S enters as a point of the inner boundary is finite, and either (2) the "projected area" of S on the coordinate hyperplanes is finite or (2′) every point is accessible in a certain sense from without, save at a set (π'), there exists a unique function, satisfying the given boundary values save at (π) and (π').

This harmonic function appears as the double limit of any sequences φ for which the Dirichlet integrals of the corresponding set V, taken over any inner region, approach their lower bound for that region. The method applies also to infinite regions.

35. In the *Transactions*, volume 9, page 373, Professor Birkhoff has shown the general character of the expansion of an arbitrary function $f(x)$ in terms of the characteristic solutions of a certain linear differential system of the nth order, and has proved the convergence of the expansion. In the present paper Dr. Milne studies the degree of convergence of the same expansion, and shows that when $f(x)$ and its first $m-1$ derivatives vanish at both ends of the interval the remainder after ν terms of the expansion will be less than a constant multiple of $1/\nu^m$ if $f^{(m)}(x)$ is continuous and of limited variation, and less than a constant multiple of $\log \nu/\nu^{m+1}$ if $f^{(m)}(x)$ satisfies a Lipschitz condition.

36. Professor Evans carries out, by means of curvilinear coordinates, a generalization to any curve of a method of Bôcher which was based on the circle. By means of a double integration, regarded as an iterated integration with respect to the curvilinear coordinates, formulas are obtained for solutions of Laplace's and Poisson's equations in terms of the boundary values on an arbitrary curve. The normal derivative of the Green's function thus appears as a ratio of two quantities expressible in terms of the curvilinear coordinates, and therefore can be calculated graphically whenever these can be drawn. Green's theorem is not used in the determination of the expressions for the solutions.

The method possesses the advantage that it applies directly to three dimensions as well as to two.

37. Mr. Alexander gives a simple proof that every Cremona plane transformation is the product of quadratic transformations. The paper will appear in the *Transactions*.

38. In the *American Journal of Mathematics* (1913) Professor Dines published a paper on the "Eliminant of several polynomials." At the suggestion of Professor Coble Mr. Robinson worked out the same problem by a direct method

analogous to the greatest common divisor process. His results were published in the last issue of the Johns Hopkins Circular (July, 1915).

F. N. COLE,
Secretary.

WINTER MEETING OF THE SOCIETY AT COLUMBUS.

THE thirty-sixth regular meeting of the Chicago Section, being the fifth regular meeting of the American Mathematical Society in the west, was held at Columbus, Ohio, on Thursday, Friday and Saturday, December 30, 31, 1915, and January 1, 1916, in affiliation with the American Association for the Advancement of Science.

About one hundred persons were in attendance upon the various sessions, including the following sixty-seven members of the Society: Professor R. B. Allen, Professor Frederick Anderegg, Professor G. N. Armstrong, Professor R. P. Baker, Professor W. H. Bates, Professor P. P. Boyd, Professor Daniel Buchanan, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Professor H. E. Cobb, Professor Elizabeth B. Cowley, Dr. L. C. Cox, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Peter Field, Professor B. F. Finkel, Professor T. M. Focke, Professor W. S. Franklin, Professor Harriet E. Glazier, Professor M. E. Graber, Professor Harris Hancock, Professor E. R. Hedrick, Dr. Cora B. Hennel, Dr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Mr. J. H. Kindle, Professor H. W. Kuhn, Professor Gertrude I. McCain, Dr. J. V. McKelvey, Dr. T. E. Mason, Professor F. E. Miller, Professor G. A. Miller, Professor J. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor C. C. Morris, Professor F. R. Moulton, Professor A. D. Pitcher, Dr. V. C. Poor, Professor S. E. Rasor, Professor H. W. Reddick, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor R. E. Root, Professor D. A. Rothrock, Professor F. H. Safford, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor K. D. Swartzel, Dr. E. H. Taylor, Mr. C. E. Van Orstrand, Professor C. A. Waldo, Professor C. J. West, Professor H. S. White, Professor E. J. Wilczynski, Professor

F. B. Wiley, Professor C. B. Williams, Professor B. F. Yanney, Professor A. E. Young, Professor Alexander Ziwet.

The opening session on Thursday afternoon was a joint meeting with Section A of the American Association, at which the program consisted of

(1) Retiring address of Professor H. S. WHITE, Vice-President of Section A, on "Poncelet polygons."

(2) Retiring address of Professor E. J. WILCZYNSKI, Chairman of the Chicago Section, "Some remarks on the historical development and the future prospects of the differential geometry of plane curves."

(3) Address by Professor W. W. CAMPBELL, of the Lick Observatory, President of the American Association, on "The rotation of planetary nebulæ."

In the absence of Professor A. O. Leuschner, vice-president of Section A, Professor C. A. Waldo was called upon to preside at this meeting. At the other sessions of the Chicago Section, the chairman, Professor E. J. Wilczynski, presided, with occasional relief by Professor H. S. White, former president of the Society.

At the business meeting of the Chicago Section on Friday afternoon, the following officers of the Section were elected for the ensuing period of two years: Professor W. B. Ford, chairman; Professor Arnold Dresden, Secretary; Professor H. L. Rietz, third member of the program committee. In presenting the nominations for these offices, Professor G. A. Miller, chairman of the committee, announced that Professor H. E. Slaught, who had served as Secretary of the Section since 1907, had expressed a desire to be relieved at this time. On Professor Miller's motion, appreciation of Professor Slaught's services was indicated by a unanimous rising vote.

Professor R. D. Carmichael reported for the retiring program committee that no arrangements had been made for a symposium at the next meeting and recommended that this question be continued in the hands of the new program committee. This recommendation was adopted.

The joint dinner of Section A and the Chicago Section on Thursday evening was held at the Ohio Union, where the arrangements and service were unusually satisfactory and highly appreciated. About seventy were present at the dinner, which was followed by an evening of enjoyable social intercourse, interspersed with some informal remarks by

Professors H. S. White, E. R. Hedrick, H. E. Slaught, and others who were called on by the chairman.

On Friday morning, the Department of Mathematics of Ohio State University tendered a luncheon to all mathematicians in attendance at the meetings, thus affording another opportunity for a social gathering and placing all guests of the occasion under renewed obligation to their hosts.

By a rising vote at the final session Saturday afternoon the members of the Society acknowledged their deep obligation to the local committee under the direction of Professor S. E. Rasor, to the Department of Mathematics of Ohio State University, and to the University as a whole, for the very careful attention which had been given to all the provisions for these meetings and for the generous hospitality which had been extended in every way throughout the period.

The following papers were presented at this meeting:

(1) Professor H. T. BURGESS: "Note on the reduction of a family of quadratic forms."

(2) Professor J. B. SHAW: "Orthogonal vector systems in vector fields of three and more dimensions."

(3) Dr. A. J. KEMPNER: "On transcendental numbers."

(4) Dr. V. C. POOR: "A certain type of exact solutions of the equations of motion of a viscous liquid."

(5) Dr. V. C. POOR: "Transformation theorems in the theory of the linear vector function."

(6) Professor A. E. YOUNG: "On the determination of a certain class of surfaces."

(7) Professor C. J. WEST: "Note on nine-fold and four-fold correlation."

(8) Professor R. D. CARMICHAEL: "On a general class of series of the form $\Sigma\, C_n g(x+n)$."

(9) Dr. H. M. SHEFFER: "On a set of independent postulates for complex algebra."

(10) Dr. H. M. SHEFFER: "Mutually prime postulates."

(11) Professor C. H. SISAM: "On surfaces doubly generated by conics."

(12) Professor G. A. BLISS: "A note on the problem of Lagrange in the calculus of variations."

(13) Professor DANIEL BUCHANAN: "Three-dimensional periodic orbits of a particle subject to the attraction of a sphere having prescribed motion."

(14) Dr. W. V. LOVITT: "A type of singular points for a transformation of three variables."

(15) Dr. W. W. KÜSTERMANN: "Functions of bounded variation."

(16) Professor T. H. HILDEBRANDT: "Green's functions connected with general linear differential equations."

(17) Professor C. N. MOORE: "On the developments in Bessel's functions."

(18) Professor E. J. WILCZYNSKI: "Integral invariants in projective geometry."

(19) Professor G. A. MILLER: "Limits of transitivity of a substitution group."

(20) Professor G. A. MILLER: "Finite groups represented by special matrices."

(21) Professor R. P. BAKER: "The four-color map theorem."

(22) Mr. W. L. HART: "Differential equations and implicit functions in infinitely many variables."

(23) Professor S. E. RASOR: "On the integration of Volterra's derivatives."

(24) Mr. A. R. SCHWEITZER: "An apparent anticipation of Hilbert's conception of completeness."

(25) Mr. A. R. SCHWEITZER: "A bifurcative generalization of a functional equation due to Cauchy."

(26) Miss IDA M. SCHOTTENFELS: "A class of functions which are self-reciprocal in the sense of Mellin."

Mr. Hart was introduced by Professors E. H. Moore and F. R. Moulton. The papers of Professors Shaw, Sisam, Bliss, and Hildebrandt, Drs. Lovitt and Küstermann, Miss Schottenfels, and the first paper of Mr. Schweitzer, were read by title.

Abstracts of the papers follow in the order indicated in the above list of titles:

1. Let $\lambda A + B$ be the matrix of a family of quadratic forms in which A is non-singular. Subject the family successively to the two linear transformations whose matrices are H and T respectively. Then

$$H'(\lambda A + B)H = H'AHH^{-1}(\lambda I + A^{-1}B)H = R(\lambda I + N),$$

$$T'R(\lambda I + N)T = T'RTT^{-1}(\lambda I + N)T = M(\lambda I + N).$$

Professor Burgess points out that the matrix H may be so

chosen that N is in the normal form and at the same time R is a matrix blocked off into principal minors each of which corresponds to an elementary divisor of $\lambda I + N$. The orders of the principal minors are the degrees of the corresponding elementary divisors; none of these principal minors overlap. He next shows that the matrix T may be so chosen that the normal form is unchanged and at the same time M differs from R in having all of its elements zero except those in the left principal diagonal of each of its blocks. As a consequence, all the well-known theorems on the elementary divisors of $\lambda A + B$ follow without further demonstration. The method of determining H and T is extremely simple and practical.

2. Professor Shaw's paper is a consideration of systems of three or more mutually orthogonal vectors in a field, such as the tangent, normal, and binormal of the vector lines, the normal and tangents of lines of curvature for a system of surfaces, and similar figures for polydimensional space. A linear vector operator of fundamental importance and much utility not hitherto noticed is exhibited, which for three-dimensional space is, in quaternion notation, α, β, γ being the moving orthogonal unit vectors,

$$\theta() = \rho() - (V\nabla\alpha)S\alpha - (V\nabla\beta)S\beta - (V\nabla\gamma)S\gamma,$$

where $2\rho = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$.

In terms of this operator the vector differential of any one of the three, in any one of the three directions is given by

$$(-S\lambda\nabla)\mu = V\mu\theta(\lambda).$$

The operator, of long-known use, $-S()\nabla\cdot\mu$, is given by

$$-S()\nabla\cdot\mu = V\mu\theta().$$

Application is made to congruences of lines.

3. Liouville (*Journal de Mathématiques*, volume 16, 1851) proved that $\sum_{r=0}^{r=\infty}\alpha_r/a^{m_r}$, where the α_r are real integers limited in absolute value and a a real integer $\geqq 2$, represents a transcendental number when the positive integral exponents $m_0, m_1, m_2, \cdots$ increase with sufficient rapidity. It is not

known, however, what constitutes this sufficient rapidity, so that the theorem does not serve to decide the transcendency of a number given in the form $\sum_{r=0}^{r=\infty}\alpha_r/a^{m_r}$ except in some extreme cases (including for example $m_r = r!, m_r = r^r$ (see Faber, *Mathematische Annalen*, volume 58, 1904).

Dr. Kempner proves that the power series

$$\sum_{r=0}^{\infty}\frac{\alpha_r}{a^{c^r}}\cdot x^r$$

has a transcendental value for every real rational value of x, if a and c are real integers $\geqq 2$, and if certain conditions are imposed on the integers α_r. These conditions are amply satisfied when: (1) $|\alpha_r| < M^r$, M arbitrary but fixed, and (2) only a finite number of the $\alpha_r = 0$.

4. The difficulty in obtaining exact solutions of the differential equations of a viscous liquid is due to their quadratic character. The most recent work of this kind has been done by C. W. Oseen. (See *Arkiv för Mathematik, Astronomi och Fysik*, volumes 3, 4, 6, 7, 9. Also *Acta Mathematica*, volume 34, 1911.) In the present paper, Dr. Poor proves the existence of a solution of the differential equations of a viscous liquid for all positive values of the time and in an infinite region. The body forces are assumed to be of the particular form

$$F(x, y, z, t) = F^{(1)}(x, y, z)e^{-k^2t}.$$

The method used is one of successive approximations. The successive steps involve solutions of sets of linear partial differential equations, the existence of which solutions is proved. Finally the convergence of the process is proved by using a dominance property of the successive steps. This dominance property persists if $F^{(1)}$ is properly restricted. Vector methods are used throughout the analysis.

5. Dr. Poor's second paper appeared in full in the January BULLETIN.

6. In two previous papers, Professor Young has discussed the problem of determining various classes of surfaces, taking as the starting point the fundamental equations written in the form first suggested and used by Bonnet in similar work. In

the present paper, he discusses from the same standpoint the general class of surfaces characterized by having $D = \pm D''$, using the customary notation, the lines of reference being lines of curvature.

7. Statistical data in the social sciences frequently cannot be classified into more than two or three broad classes. Professor West derives the working formulas for the application of the method of the correlation ratio to this type of correlation problems and discusses the value of the method as compared with certain other methods that have been proposed.

8. The series treated by Professor Carmichael are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n), \qquad \overline{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where $g(x)$ is a function having the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right)$$

valid for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals, the functions $P(x)$ and $Q(x)$ being polynomials. For the special case in which $g(x) = 1/\Gamma(x)$ the series $(\overline{\Omega}x)$ is a factorial series. Professor Carmichael points out that the series $\Omega(x)$ and $\overline{\Omega}(x)$ are of great importance in investigating the properties of functions in the neighborhood of singularities of certain frequently occurring sorts and indicates his purpose to devote several memoirs to the development of a general theory of these series, especially in relation to the function-theoretic problem mentioned. In the present paper the foundation of a general theory of these series is laid. In case the series $\Omega(x)$ neither converges everywhere nor diverges everywhere its region of convergence [absolute convergence] is the half-plane $R(\sigma x) \leqq \lambda$ [$R(\sigma x) \leqq \mu$], where $\lambda[\mu]$ is a constant depending on $c_0, c_1, c_2, \cdots$ and σ is the coefficient of the leading term in $Q(x)$ or in $P(x)$ according as $Q(x)$ is or is not of greater degree than $P(x)$. For the values of λ and μ explicit formulas are given analogous to the Cauchy-Hadamard formula for the radius of convergence of a power series. The paper contains

also a treatment of uniform convergence, of the existence of singularities of the sum-function on the boundary of the region of convergence and of uniqueness of expansions in these series. A given function $f(x)$ has not more than one expansion $\Omega(x)$ or $\overline{\Omega}(x)$ when $g(x)$ is given and $R(\sigma) > 0$. For use in investigating this theory it was found convenient to employ a certain interesting generalization of the "generalized Dirichlet series"; and the theory of these generalized series was developed to the extent needed for the applications in question.

9. The postulate set for ordinary complex algebra presented by Dr. Sheffer is based on an undefined class and three undefined operations. The set differs from previous sets in that: (1) the number of undefined entities and of postulates is reduced; (2) the order relation is defined; and (3) the existence of 0, 1, negatives, reciprocals, and imaginaries is proved.

10. A set of m postulates such that no $m - 1$ of the postulates imply the mth is called independent. If P and Q are any two postulates of an independent set, P does not imply the whole of Q. P may imply, however, a part of Q. Two postulates, neither of which implies any part of the other, may be called mutually prime; and a set of postulates which are mutually prime by pairs may be called a set of mutually prime postulates. Obviously, mutual primeness implies independence; but not conversely. Also, mutual primeness implies E. H. Moore's complete independence; but not conversely. Dr. Sheffer shows how to construct sets of postulates which are mutually prime.

11. In this paper, Professor Sisam determines some fundamental properties of the surfaces of order seven and eight which contain two pencils of conics.

12. Professor Bliss's paper appeared in full in the February BULLETIN.

13. In an article entitled "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies" (*Transactions*, volume 8 (1907), pages 159–188), Longley discussed the periodic motion of a particle which moves subject to the Newtonian attraction of n finite bodies having pre-

scribed motion. The finite bodies and the particle are restricted to move in the one plane and the coordinates of the finite bodies, when referred to one of the bodies as origin, are assumed to be known functions of the time. In the present paper Professor Buchanan discusses the periodic motion of the particle when the finite bodies have the motion prescribed in Longley's article, but the particle is not restricted to move in a plane. Periodic solutions are determined as power series in a certain parameter which may be expressed as a function of the initial projection from the plane of motion of the finite bodies. These solutions have a period commensurable with the period of Longley's solutions and reduce to the latter when the initial projection from the plane of motion becomes zero.

14. Dr. Lovitt's paper appeared in full in the February BULLETIN.

15. The idea of a function of bounded variation, first developed for functions of one variable by Jordan, has been extended to two variables by Arzelà and Hardy in papers published in 1905. These authors' generalizations differ definitionally. Dr. Küstermann asks whether they coincide conceptually and shows that they do not by constructing a function which, while monotonically increasing in both x and y, and hence of bounded variation in Arzelà's sense, is not so according to Hardy's definition. Since both types of functions are integrable and can be developed into a double Fourier series it thus appears that Arzelà's generalization is not only the broader, but also the more natural one, preserving more closely the analogy to functions of a single variable. Nevertheless, in recent papers, Lebesgue and W. H. Young, apparently unacquainted with Arzelà's work, are using Hardy's definition. At any rate the designation "function of bounded variation in two variables" is not unique, but applies to-day to two distinctly different classes of functions.

16. The paper of Professor Hildebrandt is a generalization, in the sense of Moore's general analysis, of the memoir by Schlesinger: "Zur Theorie der linearen Integralgleichungen"*

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 24, pp. 84–123.

and that of Bounitzky: "Sur la fonction de Green des équations différentielles linéares ordinaires."* The differential equations treated are of the form:

$$(1)\quad \frac{d}{dx}\eta(p'p''x) - \alpha(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

$$(2)\quad \frac{d}{dx}\eta(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

where p' and p'' range over two general classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, x ranges over a bounded linear interval $x_0 \leqq x \leqq x_1$, α and η are suitably conditioned functions of these variables, and J is a linear operator on functions of the type $\kappa(q''q')$. The first part of the paper contains existence theorems, general solutions of the equations (1) and (2), a consideration of the analogue of the Fredholm determinant for the solutions, and a Green's theorem relating to the right-hand members of equations (1) and (2), and suitably defined adjoint expressions. The second part is devoted to the consideration of a system of Green's functions $\Gamma(p'p''x, y)$ which are continuous in x and y except for $x = y$, where the discontinuity is a function of the form $\kappa(p'p'')$. These Green's functions satisfy equations of the form (2), and certain general boundary conditions. The usual theorems relating to such functions are obtained, as well as the solutions of non-homogeneous equations of the form

$$\frac{d}{dx}\eta(p'p''x) = \alpha_0(p'p''x) + J_{q''q'}\alpha(p'q''x)\eta(q'p''x),$$

satisfying the given initial conditions.

17. There are certain methods of establishing the convergence of the developments in Bessel's functions that leave unsettled the question of the value to which they converge. One of these methods is the only means thus far available of establishing the convergence at the origin of the developments in Bessel's functions of order zero for the case where the function developed has discontinuities.† Hence, in order to have a complete treatment of the developments in question

* *Journal de Mathématiques*, ser. 6, vol. 5 (1909), pp. 65–125.
† Cf. *Transactions*, vol. 12 (1911), p. 181.

that is adequate for the applications to mathematical physics and at the same time is not unnecessarily cumbersome, it is desirable to have a simple proof of the fact that the convergence is to the desired value. In Professor Moore's paper such a proof is given.

18. The simplest integral invariant of the projective theory of plane curves may be written in the form

$$H = \int \sqrt[3]{\theta_3}\, dx, \tag{1}$$

where

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{2}$$

is the differential equation of the curve, and where

$$P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'',$$

$$\theta_3 = P_3 - \tfrac{3}{2}P_2'.$$

Any other integral invariant of the curve may be expressed in the form $\int I dH$, where I is any one of its absolute differential invariants.

The integral H is as fundamental in the projective theory of plane curves as the length of arc, the fundamental integral invariant of the metric theory, is for metric geometry. The purpose of Professor Wilczynski's paper is to call attention to this fact and to provide a geometric interpretation for the integral H. His interpretation is as follows:

Consider any arc of the curve corresponding to the interval $a \leqq x \leqq b$ of the independent variable. Divide this interval into n parts by means of the values $x_0 = a, x_1, x_2, \cdots x_{n-1}, x_n = b$ such that $\lim \delta x_k = \lim (x_{k+1} - x_k) = 0$ as n grows beyond bound. Let $A, P_1, P_2, \cdots P_{n-1}, B$ be the points on the curve which correspond to these $n + 1$ values of x. Let t_k be the tangent and C_k the eight-pointic nodal cubic of P_k. The three points of inflection of the cubic C_k are on a line i_k which intersects t_k in a point I_k. Denote by τ_k one of the inflectional tangents of C_k and let T_k be its intersection with t_k. The line P_kP_{k+1} will intersect i_k and t_k in two points I_k' and T_k', and the cross-ratio $(I_k', T_k', P_k, P_{k+1})$ turns out to be equal to

$$1 - \frac{3}{\sqrt[3]{10}} \sqrt[3]{\theta_3(x_k)}\, \delta x_k.$$

By a perspective correspondence the three points I'_{n-1}, T'_{n-1}, P_{n-1} of $P_{n-1}B$ may be projected into the points I'_{n-2}, T'_{n-2}, P_{n-1} of $P_{n-2}P_{n-1}$. Let B_{n-1} be the point of $P_{n-2}P_{n-1}$ which in this perspective corresponds to B. Then project similarly I'_{n-2}, T'_{n-2}, P_{n-2}, B_{n-1} into the four points I'_{n-3}, T'_{n-3}, P_{n-2}, B_{n-2} of $P_{n-3}P_{n-2}$, and continue in this way. We shall finally obtain upon the line AP_1 a point B_1 determined from B by this sequence of perspectives. As n grows beyond bound, B_1 will approach a limiting position Q on the initial tangent t_0 of the arc AB. Let κ denote the double-ratio $\kappa = (I_0, T_0, A, Q)$. We shall have

$$\log \kappa = \frac{3}{\sqrt[5]{10}} \int_a^b \sqrt{\theta_3}\, dx,$$

which gives the desired interpretation.

It is also easy to write down projective integral invariants in the theory of space curves and surfaces. None of these however have as yet received any interpretation.

19. On page 68 of this volume of the BULLETIN Professor Miller established the theorem that a substitution group of degree n which is neither alternating nor symmetric cannot be more than $3\sqrt{n} - 2$ times transitive when $n > 12$. In the present note he points out that it results from the main theorem proved in the article mentioned that such a group cannot be more than $\frac{5}{2}\sqrt{n} - 1$ times transitive. For large values of n this evidently gives a much smaller upper limit for the degree of transitivity than the one mentioned above, which is itself much smaller than the one commonly given, namely, $\frac{1}{2}n + 1$.

20. The direct object of Professor Miller's second paper is to prove that every finite group which contains an abelian subgroup of half its order can be represented by square matrices all of whose elements are equal to zero, with the exception of those which appear in one of the diagonals, and all of these are ordinary complex numbers which are different from zero. Incidentally several other somewhat general theorems are established. Among these are the following:

If G is an abelian group of order p^m and if H is a subgroup of order p^s, then a set of independent generators of G can

always be so selected that at most $m - \alpha$ of them are not contained in H whenever G contains more than $m - \alpha$ independent generators. Moreover, it is always possible to construct a group having k independent generators and an arbitrary quotient group of order p^α such that the subgroup corresponding to identity of this quotient group cannot involve more than $k - \alpha$ of the operators in any possible set of independent generators of the group. If t transforms an abelian group G of order 2^m according to an automorphism of order 2 and if H is the subgroup of G formed by its operators which are invariant under t, then a set of independent generators of G can be so chosen that H contains all of them with the exception of at most three for each invariant of G/H.

21. The problem of the four-color map is taken by Professor Baker in the dual form. Every polyhedral net on the sphere can have its vertices marked with four colors so that no edge has the same color at its ends. The net is prepared and the problem reduced to that of an all-triangle polyhedron without triple circuits. This class is shown to be traversable by a closed curve passing once through all the vertices and having any pair of connected edges on the contour. For such a configuration a cardinal number relation is obtained for the number of successful colorings which is used as the basis of a two-step descending induction. The corresponding problem for one-sided closed surfaces of connectivity 2, 3, 4 is also solved. The numbers are 6, 6, 7.

22. At the April (1915) meeting of the Chicago Section, Mr. Hart presented a preliminary report which dealt with a part of the results of his present paper.

This investigation is concerned with functions f of the real variable $\xi = (x_1, x_2, \cdots)$ in the space

$$R: \quad M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \qquad (M_1^{(i)}, M_2^{(i)} \leqq M; i = 1, 2, \cdots).$$

A function f is said to be completely continuous at the point ξ of R if, whenever

$$\lim_{n=\infty} x_{in} = x_i \qquad (i = 1, 2, \cdots),$$

it follows that

$$\lim f(\xi_n) = f(\xi) \qquad (\xi_n = x_{1n}, x_{2n}, \cdots).$$

The results obtained are of three sorts. In the first place, theorems on completely continuous functions are derived which include, for example, the Weierstrass theorem on uniformly convergent sequences of continuous functions and Taylor's theorem. In a second part of the paper the fundamental theorem of implicit function theory is proved for the infinite system of equations

$$(1) \qquad f_i(\xi, \eta) = 0 \quad [\xi \text{ in } R; \eta = (y_1, y_2, \cdots); \eta \text{ in } R],$$

which defines ξ as a function of $(y_1, y_2, \cdots)$. Then, finally, there is considered the infinite system of ordinary differential equations

$$(2) \qquad \frac{dx_i}{dt} = f_i(\xi, t)$$

$$(i = 1, 2, \cdots; \; x_i(t_0) = a_i; \; \xi \text{ in } R; \; |t - t_0| \leqq r_0),$$

and, under suitable hypotheses, the existence of a unique continuous solution $\xi(t)$ is established.

In the second and third parts of the paper the existence proofs are constructed by methods of successive approximation in which the theorems of the first section are of fundamental importance. The classical existence theorems for finite systems of implicit functions and differential equations are special cases of the results for systems (1) and (2).

23. In the *Rendiconti dei Lincei*, Volterra defined functions of lines, their continuity, and their derivatives. In later publications, he gave a method by means of Stokes' theorem for finding anti-derivatives for these functions of a line. The object of Professor Rasor's paper is to point out an instance quite analogous to the above from the calculus of variations, using Euler's equation for this purpose.

24. The object of Mr. Schweitzer's note is to call attention to Kempe's "law of continuity" which he phrases* as follows: "No entity is absent which can consistently be present." Kempe is careful to remark that the function of this law is to ensure the "complete definition" of his system (l. c., page 149) and that the law applies to "geometric sets" (page 177).

* *Proceedings London Math. Soc.*, vol. 21.

It appears* that Kempe's statement may desirably replace Hilbert's "axiom of completeness." Both principles emphasize the interdependence of mathematics and psychology and the problem presented by the relativity of the principles as used by their respective authors seems worthy of careful study.

25. Equivalent to Cauchy's well-known functional equation $\lambda(x+y)=\lambda(x)+\lambda(y)$ are the equations

$$(1)\qquad \lambda(x-z)-\lambda(x-y)=\lambda(y-z),$$

$$(2)\qquad \lambda(z)+\lambda(x+y)=\lambda(x)+\lambda(y+z)$$
$$=\lambda(y)+\lambda(x+z),\quad \lambda(0)=0.$$

In Mr. Schweitzer's second paper an interesting "bifurcation" is represented by the following generalizations:

$$(1')\quad f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \cdots, \lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\}$$
$$=\mu f(x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}),\quad i_k=1, 2, \cdots, (n+1),$$

$$(2')\quad \mu_i\phi\{\lambda_1(x_1), \cdots, \lambda_j(x_{i-1}), \lambda_{j+1}(x_{i+1}), \cdots, \lambda_n(x_{n+1}),$$
$$\lambda_{n+1}\phi(x_i, t_1, t_2, \cdots, t_n)\}$$
$$=\phi\{\lambda_1(t_1), \lambda_2(t_2), \cdots, \lambda_n(t_n), \lambda_{n+1}\phi(x_1, x_2, \cdots, x_{n+1})\}$$

where in the latter† system $i=1, 2, 3, \cdots, (n+1)$, $x_0\equiv x_2$, $x_{n+2}\equiv x_n$. Equations (1') have been previously discussed by the author. In the case of equations (2'), the specialization $\lambda_1(x)=\lambda_2(x)=\cdots=\lambda_{n+1}(x)=\mu_i(x)=x$ leads to functional equations discussed by Abel, Stäckel, and Hayashi.

26. In the *Mathematische Annalen*, volume 68 (1910), pages 314–326, Mellin states the two following reciprocity theorems:

I. If $F(x)$ is any function belonging to a properly defined class and if

$$\varphi(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(x)t^{-x}dx,$$

* Cf. the "axiom of completeness" of G. Rabinovitch, BULLETIN, vol. 12 (1905–1906), p. 433 (abstract): "no motion is impossible unless it contradicts the above axioms."

† When $i=n+1$, the argument of λ_n is x_n.

then

$$F(x) = \int_0^\infty \varphi(t)t^{x-1}dt.$$

II. If $\varphi(x)$ satisfies the conditions in Theorem I, and if $F(t) = \int_0^\infty \varphi(x)x^{t-1}dx$, then reciprocally,

$$\varphi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(t)x^{-t}dt.$$

Example:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt,$$

$$e^{-t} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(x)t^{-x}dx \text{ for } a > 0;\ -\frac{\pi}{2} < \arg.\ t < \frac{\pi}{2}.$$

Miss Schottenfels' paper treats of a class of functions which are self-reciprocal in the above sense of reciprocity.

H. E. SLAUGHT,
Secretary of the Chicago Section.

ON PIERPONT'S DEFINITION OF INTEGRALS.

BY PROFESSOR M. FRÉCHET.

(Read before the American Mathematical Society, December 27, 1915.)

IN the second volume of his Lectures on the Theory of Functions of Real Variables, Professor J. Pierpont has given a new definition of Lebesgue integrals. This definition is interesting in as much as it realizes an effort to adapt the previous methods of presentation of Riemann integrals to the newer Lebesgue integrals.

But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function $f(x)$ on any linear set E_μ. Such definitions should not, of course, be arbitrary ones, and there are some primary conditions to be fulfilled, unless these definitions are to become quite artificial and uninteresting.

For instance, it is to be expected that for any $f(x)$ and any E,

$$\overline{\int_E} f \geqq \underline{\int_E} f,$$

if the left and right sides denote the upper and lower integrals of $f(x)$ over E. And when $f \equiv 1$ these integrals ought to reduce to the upper and lower measures of E. However, if $f \equiv 1$ it will be found that those integrals are respectively the lower and upper bounds of Σ $\overline{\text{meas.}}$ δ_n, where $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ is a "separated division of A into cells."* Hence

$$\overline{\int_E} f \leqq \underline{\int_E} f;$$

and the equality cannot hold for every E. For if, for instance, E is the interval (0, 1) and if δ_1 is a non-measurable part of E and $\delta_2 = E - \delta_1$, then $\overline{\text{meas.}}$ $\delta_1 + \overline{\text{meas.}}$ $\delta_2 >$ meas. E. But of all the separated divisions of E, take those two the first of which consists of E itself and the second of (δ_1, δ_2); then

$$\overline{\int_E} f \leqq \text{meas. } E,$$

$$\underline{\int_E} f \geqq \overline{\text{meas.}}\ \delta_1 + \overline{\text{meas.}}\ \delta_2.$$

Thus we get a case where at the same time

$$\overline{\int_E} f < \underline{\int_E} f$$

and, though $f = 1$,

$$\underline{\int_E} f > \text{meas. } E.$$

Curiously enough, Professor Pierpont did not think it useful to mention that the inequality

$$\overline{\int_E} f - \underline{\int_E} f \geqq 0$$

* δ_i, δ_k are said to be separated when they are enclosed respectively in two measurable sets whose common part has the measure zero. Here $\overline{\text{meas.}}$ means upper measure.

should hold, as should result from his relation

$$\overline{\int_E} f - \underline{\int_E} f = \lim. \Sigma \omega_n \times \overline{\text{meas.}}\ \delta_n,$$

where ω_n is the oscillation of f in δ_n. However, it results from my example above that this relation is not always true.

The mainspring of all these difficulties is the error made in theorem 376, page 369: "Let $A = (B, C)$ be a separated division of A, then meas. A = meas. B + meas. C," which is a generalization of theorem 341, page 346, "If $A = B + C$ and B, C are exterior to each other, meas. A = meas. B + meas. C." The assumption made in the second line of the proof of this theorem is not altogether obvious, so that the proof is not convincing. Moreover, the theorem itself does not hold in every case; for instance, it does not when A is an interval and B a non-measurable part of A.

It is further found that the demonstration of the inequalities

$$(1) \qquad m \times \overline{\text{meas.}}\ E \leqq \underline{\int_E} f; \qquad \overline{\int_E} f \leqq M \times \overline{\text{meas.}}\ E$$

is based explicitly (§ 379, page 372) on the first theorem reproduced above and on a consequence of it which reads as follows:

$$m \times \overline{\text{meas.}}\ E \leqq \underline{S}_D; \ \overline{S}_D \leqq M \times \overline{\text{meas.}}\ E.$$

Now this consequence is easily seen to be false itself, whereas the final inequalities (1), which are correct, would have been more easily proved by showing that $m \times \overline{\text{meas.}}\ E$ and $M \times \overline{\text{meas.}}\ E$ are particular values assumed by $\underline{S}_D$ and $\overline{S}_D$ when D consists of E alone.

At any rate, many difficulties should disappear if the δ_n are to be measurable. No doubt E would then itself be measurable and the definition would not have so large an extent. However the case of the non-measurable E—which is not particularly interesting—may be easily dealt with by enclosing E in any measurable set B, letting $f = 0$ in $B - E$ and putting

$$\underline{\int_E} f = \underline{\int_B} f, \qquad \overline{\int_E} f = \overline{\int_B} f,$$

these values being obviously independent of the choice of B.

Finally, I fail to see any advantage in the use of the so-called separated divisions of E. The results are exactly the same and a useless complication is avoided if E is only divided into parts exterior to each other.

Now divide a measurable set E into a countable sequence of measurable subsets δ_i exterior to each other and denote by $\overline{\int_E} f$ and $\underline{\int_E} f$ the lower and upper bounds of $\Sigma M_i\delta_i$ and $\Sigma m_i\delta_i$, where m_i, M_i are the lower and upper bounds of f on δ_i. By these definitions, the upper integral is never smaller than the lower integral. And if $f \equiv 1$, both integrals are equal to meas. E.

This new definition is very similar to that of Riemann. The real difference is *not* as Professor Pierpont asserts for his own that it makes use of an infinite instead of a finite number of parts of E. It lies essentially in the use of measurable parts of E instead of intervals. For instance when f is bounded over E, the definition is not altered if the parts δ_i of E are assumed to be in infinite (variable) number.

When $\underline{\int_E} f = \overline{\int_E} f$ the common value of both integrals is equal to the value of the corresponding Lebesgue integral.

UNIVERSITY OF POITIERS.

REPLY TO PROFESSOR FRÉCHET'S ARTICLE.

1. REPLYING to the foregoing criticism I begin by quoting. Professor Fréchet says: "But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function on any linear set E_μ. Such definitions should not of course be arbitrary ones, and there are some primary conditions to be fulfilled unless these definitions are to become quite artificial and uninteresting."

The implication that the reader will easily draw is that I have not fulfilled these primary conditions and that my theory is therefore quite artificial and uninteresting. Certainly flattering to the author.

To be historically accurate, I had no intention whatever of generalizing Lebesgue's integrals. When years ago I hit on my definition of integration, I did not know how it was related to Lebesgue's theory. I found out later that when the field of integration is measurable my integrals are identical with Lebesgue's and I have therefore called them Lebesgue integrals throughout my book. To prevent misunderstanding let me note that my definition is not restricted to a single variable as one may have gathered from the passage just quoted; this however is a minor matter. Professor Fréchet calls my definition inappropriate. Since my integral and Lebesgue's are the same when the field of integration is measurable, any defect in my integral is equally shared by Lebesgue's in this case. I infer therefore that his strictures apply only to the case where the field of integration is non-measurable.

I lay no great importance on this side of my definition. No non-measurable field has yet been studied as far as I know, and it may turn out that they have little value in the theory of point sets.

Theoretically they do present themselves in a rather awkward way in the theory of double integrals. Let $\mathfrak{A}$ be a measurable limited field whose projection is $\mathfrak{B}$ and whose cross sections are $\mathfrak{C}$. If $f(x, y)$ is limited and integrable in $\mathfrak{A}$, we would like to write down, as in the calculus,

$$(1) \qquad \int_{\mathfrak{A}} f(x, y)dxdy = \int_{\mathfrak{B}} dx \int_{\mathfrak{C}} f(x, y)dy.$$

Now it turns out that although $\mathfrak{A}$ itself is measurable, an infinite number of the sections $\mathfrak{C}$ *may not be*. If now we do not define integrals over non-measurable fields, the symbol

$$\int_{\mathfrak{C}} f(x, y)dy$$

which enters (1) is not defined and the same is true of the right side of (1). This difficulty may be turned in a variety of ways; one way is to use a definition of integration which does not depend on the measurability of the field.

This Professor W. A. Wilson did in 1909. He replaced the intricate and highly artificial reasoning of Lebesgue* by

* Cf. Lebesgue, *Ann. di Mat.*, 1902. Reproduced by Hobson, Functions of a Real Variable (1907), p. 576.

simpler and more direct methods as given in Volume II of my Real Variables.*

2. Let us now consider some of Professor Fréchet's objections in detail, it being understood that the field of integration is *non*-measurable. He says: It is to be expected that for any $f(x)$ and any E

$$\overline{\int}_E f \geqq \underline{\int}_E f. \tag{2}$$

Now Professor Fréchet thinks he has constructed an example which contradicts the relation (2), i. e., he thinks he has shown that in a certain case

$$\overline{\int}_E f < \underline{\int}_E f. \tag{3}$$

If this were true, my theory of integration for non-measurable fields would be in a sorry plight.

To establish the *false* relation (3) for a special case, Professor Fréchet divides the unit interval $E = (0, 1)$ into two parts by taking a non-measurable component δ_1 and its complement δ_2 such that

$$\overline{\text{meas}}\ \delta_1 + \overline{\text{meas}}\ \delta_2 > \text{meas}\ E. \tag{4}$$

So far so good, but Professor Fréchet now states that $\mathfrak{A} = (\delta_1, \delta_2)$ is a *separated* division, for he says: "But of all separate divisions of E take those two, the first of which consists of E itself and the second of (δ_1, δ_2)."

Professor Fréchet has been misled at this point; there is no separated division of $\mathfrak{A}$ such that (4) holds, and his example establishes not an error on my part but a carelessness of reasoning on his.

3. Professor Fréchet now attacks the correctness of the relation

$$\overline{\text{meas}}\ A = \overline{\text{meas}}\ B + \overline{\text{meas}}\ C, \tag{5}$$

where B, C is a separated division of $\mathfrak{A}$. He says:

(1) "The assumption made in the second line of the proof of this theorem is not altogether obvious, so that the proof is not convincing. (2) Moreover, the theorem itself does not

* Wilson's results were given by me in a course of two lectures delivered at Clark University in September, 1909. Cf. also a paper by Hobson, *Proceedings Lond. Math. Soc.*, vol. 8, Part 1. Issued December 23, 1909.

hold in every case; for instance, it does not when A is an interval and B is a non-measurable part of A."

As far as the writer can see, (2) is a bald statement unaccompanied by a shred of proof. A charge as serious as this certainly deserves some support on the part of the person making it.

Let us look at (1). The assumption in question is that

$$\mathfrak{E}_n = A_n + B_n + C_n \tag{6}$$

is an ϵ_n-enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultaneously.

The author made this statement without proof because in his judgment an attentive reader who had used the machinery of superposition up to this point would admit its truth as obvious. However to make it clear even to him that runs, we add the following, using the notation of my book:

Let

$$A_n = \{a_{n\kappa}\}, \quad B_n = \{b_{n\lambda}\}, \quad C_n = \{c_{n\mu}\}.$$

Each cell $a_{n\kappa}$ is a measurable point set containing points of $\mathfrak{A}$, $b_{n\lambda}$ one containing points of $\mathfrak{B}$, etc. Now if e_{nj} denote a cell of $\mathfrak{E}_n$, e_{nj} is by definition $Dv\{a_{n\kappa}, b_{n\lambda}\}$ or $Dv\{a_{n\kappa}, c_{n\mu}\}$. But any point of $\mathfrak{B}$ (or of $\mathfrak{C}$) is in some $b_{n\lambda}$ (or $c_{n\mu}$) and since $\mathfrak{B}$ (or $\mathfrak{C}$) is a part of $\mathfrak{A}$, in some $a_{n\kappa}$. Therefore $\mathfrak{E}_n = \{e_{nj}\}$ contains all points of $\mathfrak{A}$. Now $\mathfrak{E}_n$ is a part of A_n and hence meas $\mathfrak{E}_n \leqq$ meas A_n; therefore $\mathfrak{E}_n$ is an ϵ_n enclosure of $\mathfrak{A}$. Let B_n' denote those parts of B_n contained in $\mathfrak{E}_n$. Then meas $B_n' \leqq$ meas B_n. Hence B_n' is an ϵ_n enclosure of $\mathfrak{B}$. Similarly C_n' is an ϵ_n enclosure of $\mathfrak{C}$. Thus $\mathfrak{E}_n$ is simultaneously an ϵ_n enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

4. The next charge is that the relations

$$m\bar{\mathfrak{A}} \leqq \underline{S}_D, \quad \overline{S}_D \leqq M\bar{\mathfrak{A}} \tag{7}$$

are false. The proof that they are *correct* is given on page 372; as it requires but four lines, I reproduce it textually, viz.,

$$m \leqq m_i \leqq M_i \leqq M.$$

Hence

$$\Sigma m\bar{\delta}_i \leqq \Sigma m_i\bar{\delta}_i \leqq \Sigma M_i\bar{\delta}_i \leqq \Sigma M\bar{\delta}_i.$$

Thus

$$m\Sigma\bar{\delta}_i \leqq S'_D \leqq \overline{S}_D \leqq M\Sigma\bar{\delta}_i. \tag{8}$$

But by § 376, 2,

$$\Sigma\bar{\delta}_i = \bar{\mathfrak{A}}. \tag{9}$$

So far as in my book. To get (7) put (9) in (8). The only point in this simple demonstration which Professor Fréchet can attack is the relation (9). But this brings the question around again to the fundamental relation (5) which we have already discussed. Professor Fréchet is thus repeating himself.

In connection with the relation (7) I quote the following remark of Professor Fréchet. He says: "Curiously enough Professor Pierpont did not think it useful to mention that the inequality (2) should hold." I suppose I did not mention the relation (2) because I thought it too self-evident. The proof is immediate:

Let $D_1, D_2 \cdots$ be an extremal sequence as defined on page 374. Then replacing D by D_n in (8) above we have $S_{D_n} \leqq \overline{S}_{D_n}$. Let now $n \doteq \infty$, we get the relation (2) by § 383, 1.

5. Finally at the close of Professor Fréchet's article I am told by him that I do not know the real significance of my own work. "The real difference is not as Professor Pierpont asserts for his own (definition) that it makes use of an infinite instead of a finite number of parts of E (as in Riemann's definition). It lies essentially, etc. . . ."

Professor Fréchet will pardon me if I still hold to my original opinion in spite of his illuminating remarks. He has been so often wrong, as I hope I have made clear, that he may well be wrong here also.

The rest of Professor Fréchet's remarks relate to matters of taste and as de gustibus non est disputandum I refrain from entering the controversy. Professor Fréchet claims that I have erred on three counts, viz.: 1. The relation (2). 2. The relation (5). 3. The relation (7).

The only proof he has adduced is an example whose validity depends on establishing the *vital* fact that $\mathfrak{A} = (\delta_1, \delta_2)$ is a separated division of $\mathfrak{A}$. I expect he will hasten to remove this lacuna.

JAMES PIERPONT.

YALE UNIVERSITY,
January, 1916.

CARMICHAEL'S MONOGRAPHS ON BRANCHES OF THE THEORY OF NUMBERS.

The Theory of Numbers. By R. D. CARMICHAEL. New York, John Wiley and Sons, 1914. 8vo. 94 pages. Price $1.

Diophantine Analysis. By R. D. CARMICHAEL. New York, John Wiley and Sons, 1915. 8vo. 6+118 pages. Price $1.25.

THE various series of tracts or monographs on mathematics which are in course of publication in several European countries are so well known and the arguments in favor of them are so generally conceded that it is not surprising that several series of tracts have been recently begun in America. In view of the purpose of such a tract, the editor of a series naturally imposes a definite upper limit to its length. Frequently the tract relates to a very extensive field of mathematics and the problem of the selection of topics presents a serious difficulty to the author.

There is an added difficulty in the case of a tract for beginners in the theory of numbers (and the same point would apply to the case of the theory of groups): the subject is somewhat abstract and the nature of the theorems and proofs is quite different from that to which the reader is accustomed. Consequently the author of the tract on the Theory of Numbers has wisely adopted a very elementary and expansive style of presentation, even at the expense of a reduction of the number of topics treated. A like motive, combined with the desire to emphasize methods rather than results, doubtless led the author to give several proofs of Fermat's theorem and Euler's generalization, although the space used could have been utilized for the presentation of further results.

Chapter I deals (in 23 pages) with the uniqueness of factorization into primes, the greatest common divisor and least common multiple of two or more integers, the highest power of a prime which divides $n!$, and the simplest properties of prime numbers.

Chapter II devotes 7 pages to Euler's ϕ-function or indicator. Two methods of evaluating $\phi(m)$ are given in detail, and a third method is suggested.

Chapter III gives in 10 pages the formal properties of congruences, a proof that a congruence of degree n with respect

to a prime modulus has at most n (real) roots, and the simpler theorems on linear congruences.

Chapter IV treats (in 14 pages) of Fermat's theorem and its extensions and converse, its application to linear congruences and to Euler's criterion for quadratic residues; also Wilson's theorem and its converse.

Chapter V devotes 15 pages to a rather full account of the theory of primitive roots. Concerning the important function $\lambda(m)$, the maximum indicator of Cauchy, it is proved that there exist integers belonging to the exponent $\lambda(m)$ modulo m. Thus $x^{\lambda(m)} \equiv 1 \pmod{m}$ is satisfied by every integer x prime to m, while this is not true for an exponent less than $\lambda(m)$.

Chapter VI gives (in 17 pages) a brief first view of various additional topics such as the theory of quadratic residues, including a statement of the law of reciprocity; Galois imaginaries, from the intuitive point of view of Galois; rational right triangles.

In view of its great clearness and elementary character, this book will prove a boon to the general reader desirous of an introduction to the "queen of the sciences," as well as to those students of mathematics who wish to acquire quickly and easily a working knowledge of the theory of numbers sufficient for its ordinary applications in other fields of mathematics. For the latter purpose, the text should have contained numerous exercises involving quadratic residues. On the other topics treated, the exercises are numerous and well selected. Having examined the full reports from a beginner in this subject who used this text for private reading, the reviewer is confident that the text is well suited for the two classes of readers mentioned above. In view of the limited range of topics, the book would have to be supplemented by lectures if adopted for a major course in the theory of numbers, as usually presented at the universities.

The author has deviated from custom in his definition of three or more relatively prime numbers. This term is used (page 9) when the numbers have no common factor except unity. According to Dirichlet,* numbers are relatively prime only when every pair of them are relatively prime. While Carmichael is consistent in the use of his definition, a student who was reading the text occasionally made errors by falling back upon the older (and perhaps more natural) definition.

In exercise 7, page 17, the word two should be inserted before

* Vorlesungen über Zahlentheorie, ed. 4, 1894, p. 11.

"relatively prime factors." There is an evident misprint in exercise 3, page 20.

The text on Diophantine analysis is of much greater scientific importance than the book just reviewed, since there exists no other single book in any language which presents so much of the material on Diophantine equations, and certainly no earlier book which undertakes such a systematic presentation of important aspects of the theory. From the time of Pythagoras there has been an uninterrupted interest in the subject now known as Diophantine equations. In particular, there has been for three centuries a special interest in Fermat's last theorem, that perpetual challenge to mathematical combat, that impenetrable armor upon which has been shattered the lance of many a gallant trained soldier, that lure which has fascinated only to repel the uncouth advances of many a camp follower and raw recruit. If only to satisfy a reasonable curiosity, the general mathematical public is entitled to a clear exposition of the problems, methods and results achieved in an ancient subject in which the theorems appear to be so simple and yet are often so difficult to prove.

Chapter I deals with the general nature of Diophantine analysis, the lack of general methods of investigation, rational oblique and right triangles, and Fermat's method of infinite descent.

Chapter II is an introduction to the application in this subject of a principle, discussed later in this review, which is really only the theorem that, in a given algebraic domain, the norm of a product equals the product of the norms of the factors. The applications here are to Pell's equation $x^2 - Dy^2 = \sigma^2$ (especially the case $\sigma = 1$), to $x^2 + ay^2 + bu^2 + abv^2 = t^2$, to the complete solution of $x^2 + y^2 + z^2 = t^2$ in integers, to the derivation of a second solution of $x^4 + ay^4 + bz^4 = t^2$ from a given solution, and to a like result for $x^4 + ay^4 = \mu^2 + bv^2$.

Chapters III and IV are devoted to Diophantine equations of the third and fourth degrees in two or more variables.

Chapter V devotes 19 pages to Fermat's last theorem: If n is an integer greater than 2, there are no integers x, y, z all different from zero, such that $x^n + y^n = z^n$. The formulas given by Abel and Legendre are proved. There is a presentation of the method of Sophie Germain as developed by

Legendre and recent writers. Four pages are used to give a summary of further results known about Fermat's last theorem, many of the results being stated in the form of exercises.

Chapter VI directs attention to the possibility of using rational solutions of functional equations as a means of classifying isolated problems on Diophantine equations. The illustration employed is the functional equation

$$(a^2 + 1)(u_a^2 + 1) = v_a^2 + 1.$$

Various solutions of this are employed in a treatment of Fermat's problem to find three squares such that the product of any two of which, added to the sum of those two, gives a square.

The chief aim of the author is set forth in the preface as follows. "The task of the author has been to systematize, as far as possible, a large number of isolated investigations and to organize the fragmentary results into a connected body of doctrine. The principal single organizing idea here used and not previously developed systematically in the literature is that connected with the notion of a multiplicative domain introduced in Chapter II" (that of applying the multiplicative property of norms of algebraic numbers). Again on page 50, the author says "The method of extending this set (of numbers $x_1^n + a_1x_1^{n-1}x_2 + \cdots + a_nx_2^n$) so that the resulting set shall form a domain closed with respect to multiplication grows out of a remark due to Lagrange (Oeuvres, 7, pages 164–179), though Lagrange seems nowhere to have utilized it in connection with Diophantine problems. A partial use of it has been made by Legendre (Théorie des Nombres, volume 2, ed. 3, pages 134–141); but its consequences seem nowhere to have been systematically developed."

This programme has been carried out so admirably that it does not detract from the value of the work to point out that Lagrange did apply the idea to Diophantine equations and that later writers developed the theory quite systematically.

Lagrange* proved that, if a is a fixed nth root of unity, the product of two functions of the type

$$p = t + ua\sqrt[n]{A} + xa^2\sqrt[n]{A^2} + \cdots + za^{n-1}\sqrt[n]{A^{n-1}}$$

is of like form. Hence if we replace a by the different nth

* *Mém. Ac. R. Sc. Berlin*, vol. 23, 1769; Oeuvres 2, 527.

roots of unity and form the product of the functions so obtained from p, we obtain a rational function P of $t, u, \cdots, z, A$, such that the product of two functions of type P is a third function of type P. It is shown how P can be found by elimination. The theory is applied to the solution of

$$r^n - As^n = q^m. \tag{1}$$

We desire to express each factor $r - asA^{1/n}$ as an mth power p^m, where $a^m = 1$, and p is the above linear function. Then

$$p^m = T + Ua\sqrt[n]{A} + Xa^2\sqrt[n]{A^2} + \cdots + Za^{n-1}\sqrt[n]{A^{n-1}}.$$

Hence we take $r = T$, $s = -U$, $X = 0, \cdots, Z = 0$. Thus (1) is solvable by this method if $X = 0, \cdots, Z = 0$ are solvable. Although we have only $n - 2$ equations in n variables, they do not always have rational solutions. For the case $n = 3$, $m = 2$, the single condition $X = 0$ gives $x = -u^2/2t$; then

$$r = T = t^2 + 2Aux = t^2 - \frac{Au^3}{t},$$

$$-s = U = Ax^2 + 2tu = \frac{Au^4}{4t^2} + 2tu,$$

$$q = P = t^3 + Au^3 - 3Atux + A^2x^3.$$

For $n = m = 3$, the condition is $tu^2 + t^2v = Auv^2$; but Lagrange did not complete the discussion of this case.

The method just applied to the two cases having $n = 3$ was later extended by Lagrange* from the special case $a^3 = 1$ to the case in which a is a root of any cubic equation. This work is reproduced by Carmichael on pages 55, 56, where a reference to Lagrange would have been in place. For, although this reference was given five pages earlier, it was there stated (see quotation above) that Lagrange had not applied the idea to Diophantine problems. Carmichael reproduced Lagrange's use of the idea to obtain a set of solutions, involving two parameters, of

$$x^3 + ax^2y + bxy^2 + cy^3 = v^2. \tag{2}$$

Lagrange remarked that his solution of (2) "is well worthy of notice on account of its generality and the manner in which

* Addition IX to Euler's Algebra, vol. 2, 1774, pp. 644–9; Oeuvres de Lagrange, vol. 7, pp. 170–9.

it was derived, which is perhaps the only way which can lead to it easily."

Carmichael attempts to apply the idea to the similar equation in which v^2 is replaced by v^3, but finds (page 58) that the condition $X = 0$ is so complicated that a complete solution is hardly to be expected; he then gives the recent methods by Schaewen, based on other principles.

Lagrange* made much use of the property

$$(p^2 - Bq^2)(p_1^2 - Bq_1^2) = (pp_1 \pm Bqq_1)^2 - B(pq_1 \pm qp_1)^2$$

in his various investigations on Diophantine equations of the second degree, especially in his work on Pell's equation and in the solution of $u^2 - Bt^2 = A$ in rational numbers. The corresponding formula (page 525) concerning

$$F = p^2 - Bq^2 - Cr^2 + BCs^2$$

was used by him in the proof that every number is expressible as a sum of four squares. G. Libri† used this property of F and gave a formula stated to give all the ways of reducing a product FF_1 of two such functions to a like form F_2; he gave (page 292) an identity expressing the product of two sums of four cubes as a sum of cubes of four rational expressions, and remarked that also

$$3x^4 + y^4 - z^4 - 3u^4$$

repeats under multiplication and represents rationally all rational numbers.

Lagrange‡ stated that "the simplest and most general method for equations like $x^4 + ay^4 = z^2$ is perhaps that by factors in his additions (final chapter) to Euler's algebra."

It is therefore clear that Lagrange was fully aware of the applications of the multiplicative property of norms to Diophantine problems. Moreover, it is rather evident that he developed this property of norms for the purpose of applying it to Diophantine equations.

As to Carmichael's remark that the consequences of this idea of Lagrange's "seem nowhere to have been systematically developed," it should be noted that the series of papers by

* Oeuvres, 2, p. 386, p. 523.
† *Journal für Math.*, vol. 9, 1832, p. 287.
‡ Oeuvres, 4, p. 395.

Desboves* give such a systematic development, perhaps as extensive as Carmichael's. Among the equations treated by this method by Desboves are

$$X^3 + rY^3 = Z^2, Z^3, Z^4; \quad \xi^4 + k\eta^4 = \zeta^2.$$

In this connection should be cited a general theorem of importance due to Dirichlet:† If at least one of the roots $\alpha, \beta, \cdots, \omega$ of an equation $s^n + as^{n-1} + \cdots + h = 0$ is real and if $a, \cdots, h$ are integers, while $s^n + \cdots$ has no rational divisor, and if

$$\phi(\alpha) = x + \alpha y + \cdots + \alpha^{n-1}z,$$

then the Diophantine equation

$$F(x, y, \cdots, z) \equiv \phi(\alpha)\phi(\beta) \cdots \phi(\omega) = 1$$

has an infinity of integral solutions. If the corresponding Lagrangian function P can assume a given value N, it takes the same value N for an infinitude of sets of values of $x, y, \cdots, z$. Poincaré‡ noted that, under the same conditions, the solution of $F = N$ reduces to the problem of forming all complex ideals of norm N, and discussed the latter question.

Carmichael (pages 44–48) has applied Lagrange's method to two new types of Diophantine equations

$$x^4 + ay^4 + bz^4 = t^2, \quad x^4 + ay^4 = \mu^2 + b\nu^2,$$

obtaining a second set of solutions from a given set. When applied to another equation (page 62), the method led to a solution which "unfortunately lacks generality," so that a special method was devised. The author himself of course recognizes that Lagrange's method is not a universal panacea.

If the strict limitations of space had not made it necessary for the author to develop a considerable body of classic results from a single central point of view, he would doubtless have given an exposition of other methods of considerable generality. For example, the elegant method in the joint paper§ by Hilbert and Hurwitz to obtain all sets of rational solutions of

* *Nouv. Ann. Math.*, ser. 2, vol. 18, 1879, pp. 265–279, 398–410, 433–444, 481–499.

† *Comptes Rendus*, Paris, vol. 10, 1840, pp. 285–8.

‡ Ibid., vol. 92, 1881, p. 777.

§ *Acta Mathematica*, vol. 14, 1890–1, pp. 217–224.

$f(x, y, z)=0$, where f is a homogeneous polynomial of degree n with integral coefficients such that the curve $f = 0$ is of genus (or deficiency or Geschlecht) zero. Again, Poincaré,* with the aim to find a bond between various problems of Diophantine analysis, has treated homogeneous polynomials $f(x, y, z)$ with integral coefficients from the standpoint of classes of curves $f = 0$ under birational transformations with rational coefficients. Finally, C. Runge† and E. Maillet‡ have given conditions for an infinitude of sets of solutions of any Diophantine equation in two variables. References to papers of this type would have been in place in such a brief text.

The reference (page 68) to Euler alone is rather generous, as Euler, in his proof of the impossibility of integral solutions, all different from zero, of $x^3 + y^3 = z^3$, did not give a rigorous proof of the vital point that, if $p^2 + 3q^2$ is a cube, it is the cube of a number $t^2 + 3u^2$ and that $p + q\sqrt{-3}$ is the cube of $t + u\sqrt{-3}$. For a proof, see Pepin.§

The numerous exercises are of three types with distinguishing marks. There are 133 exercises intended to develop facility in the handling of the subject; 53 additional exercises are of more difficulty and are intended primarily as a summary of known results not otherwise included in the text; while 35 further exercises are intended to suggest investigations. With many of the exercises are affixed names of writers and dates, but no journal references. The author evidently made a thorough examination of the extensive literature of the subject.

In view of its undoubted scientific merits, the book should be very useful to the student of Diophantine analysis. In view of its clear and attractive style of presentation, its emphasis on important results and methods, and its proper subordination of minor or more technical matters, the text should appeal strongly to the general reader desirous of obtaining in a brief time a clear view of this attractive branch of the theory of numbers.

L. E. DICKSON.

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* *Jour. de Math.*, ser. 5, vol. 7 (1901), 161–233.
† *Jour. für Math.*, vol. 100 (1887), pp. 425–435.
‡ *Jour. de Math.*, ser. 5, vol. 6 (1900), pp. 261–277.
§ *Jour. de Math.*, ser. 3, vol. 1 (1875), p. 317.

NOTES.

THE opening (January) number of volume 38 of the *American Journal of Mathematics* contains the following papers: "The oscillation of functions of an orthogonal set," by O. D. KELLOGG; "On some properties of the medians of closed continuous curves formed by analytic arcs," by ARNOLD EMCH; "Theorems on the groups of isomorphisms of certain groups," by L. C. MATHEWSON; "Self-projective rational sextics," by R. M. WINGER; "On linear difference and differential equations," by C. E. LOVE; "The uniform motion of a sphere through a viscous liquid," by R. W. BURGESS; "Note on the theory of optical images," by GEORGE STEIĆ.

AT the meeting of the London mathematical society on January 13 the following papers were read: "The transition from vapor to liquid when the range of the molecular attraction is sensible," by J. LARMOR; "A note on the uniform convergence of the Fourier series $\Sigma a_n \sin n\theta$" and "A condition for the validity of Taylor's expansion," by T. W. CHAUNDY.

AT the meeting of the Edinburgh mathematical society on January 14 the following papers were read: "On the continued fractions of Tchebychef and Laguerre," by H. DATTA; "The conformal representation of the quotient of two Bessel functions," by A. MILNE.

THE following Cambridge tracts in mathematics and mathematical physics are announced as in press, to appear in a few weeks: The Definite Integral, its Meaning and Fundamental Properties, by E. W. HOBSON; An Introduction to the Theory of Attractions, by T. J. I'A. BROMWICH; Pascal's Hexagon, by H. W. RICHMOND; Lemniscate Functions, by G. B. MATHEWS; Chapters on Algebraic Geometry, by F. H. BAKER; The Integrals of Algebraic Functions, by F. H. BAKER.

DARTMOUTH COLLEGE. The following courses in mathematics will be given in the summer session, July 16 to August 16: By Professor J. W. YOUNG: The reorganization of secondary school mathematics.—By Professor E. G. BILL: Plane analytical geometry; Projective geometry.

THE University of Messina has been reopened with the following mathematical staff: V. MARTINETTI, professor of analytic geometry; Z. GIAMBELLI, formerly of the University of Cagliari, associate professor of projective and descriptive geometry; P. CALAPSO, of the University of Palermo, associate professor of algebra and analysis; E. LAURA, of the University of Turin, associate professor of mechanics and mathematical physics.

THE royal prize of 10,000 francs, conferred every six years by the Reale Accademia dei Lincei, has just been awarded to Professor FRANCESCO SEVERI, of the University of Padua, for his researches in algebraic geometry.

PROFESSOR G. H. BRYAN, of Bangor College, has been elected to an honorary fellowship in Peterhouse College, Cambridge.

PROFESSOR P. KOEBE, of the University of Jena, has been elected a corresponding member of the Göttingen academy.

PROFESSOR I. SCHUR, of the University of Bonn, has been appointed associate professor of mathematics at the University of Berlin, as successor to the late Professor KNOBLAUCH.

DR. A. FUNK has been appointed docent in mathematics at the German University of Prague.

DR. R. GRAMMEL has been appointed docent in mechanics at the technical school of Dantzig.

MR. C. GARLOUGH has been appointed instructor in mathematics at Wheaton College.

MISS S. F. RICHARDSON, assistant professor of mathematics in Vassar College, died February 2, 1916. Miss Richardson was a graduate of Vassar, and a member of the teaching staff since 1886. She became a member of the American Mathematical Society in 1905.

PROFESSOR J. W. R. DEDEKIND, of the technical school of Brunswick, died February 12, at the age of 83 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AL-KHOWARIZMI. Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi. With an introduction, critical notes and an English version by L. C. Karpinski. (University of Michigan Studies. Humanistic series, vol. 11, part 1.) New York, Macmillan, 1915. Royal 8vo. 4+164 pp.+4 plates. Paper. $2.00

BERZOLARI (L.). Geometria analitica. II: Curve e superficie del secondo ordine. (Manuali Hoepli.) Milano, Hoepli, 1916. 11+427 pp. L. 3.00

BLICHFELDT (H. F.). See MILLER (G. A.).

DICKSON (L. E.). See MILLER (G. A.).

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 3, Heft 2: C. Runge und F. A. Willers, Numerische und graphische Quadratur und Integration gewöhnlicher und partieller Differentialgleichungen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 47–176. M. 4.20

FERROLLI (T.). Geometria non-euclidea. (Biblioteca del popolo, no. 590.) Milano, casa ed. Sonzogno (Matarelli), 1915. 16mo. 61 pp. L. 0.20

FONTSERÉ Y RIBA (E.). See TALLADA (F.).

GARBE (E.). Zur Theorie der Integralgleichungen. Tübingen, 1914. 8vo. 43 pp.

GIGER (A.). Ueber die dritte Steinersche Erzeugungsweise der Fläche dritter Ordnung. Zürich, 1914. 8vo. 30 pp.

JANSSEN (G.). Ueber die definitionsmässige Einführung der affinen und der adäquaten Geometrie auf Grund der Verknüpfungsaxiome. Göttingen, 1913. 8vo. 48 pp.

KARPINSKI (L. C.). See AL-KHOWARIZMI.

KLEIN (L.). Streifzüge in das Gebiet der Mathematik und Geometrie. Heft 1: Zur Kreislehre (über Näherungskonstruktionen für algebraisch unlösbare Aufgaben aus der Kreislehre); über das sogen. Vivianische Fenster. Heft 2: Ueber die Verallgemeinerung des Feuerbachschen Kreises. Korneuburg, 1915. Gr. 8vo. 43+32 pp. M. 1.00+1.00

KODWEISS (W.). Theorie der Monge-Ampèreschen Differentialgleichung mit drei unabhängigen Variablen. Tübingen, 1913. 8vo. 48 pp.

MAZKEWITSCH (D.). Ueber projektivische Strahlen- und Punktinvolutionen und einige Erzeugnisse derselben. (Thèse, Berne.) Zürich, Müller, Werder et Cie., 1915. 8vo. 76 pp.

MEINONG (A.). Ueber Möglichkeit und Wahrscheinlichkeit. Beiträge zur Gegenstands- und Erkenntnistheorie. Leipzig, Teubner, 1915. Gr. 8vo. 16+760 pp. M. 19.00

MILLER (G. A.), BLICHFELDT (H. F.) and DICKSON (L. E.). Theory and application of finite groups. New York, Wiley, 1915. 8vo. 390 pp. $3.00

PASCAL (E.). I miei integrafi per equazioni differenziali. Napoli, Pellerano, 1915. 16mo. 137 pp. L. 6.00

PFLÜGER (G.). Die Formschönheit einfacher geometrischer Gebilde. Bausteine zu einer wissenschaftlichen Aesthetik. Stuttgart, 1915. Gr. 8vo. 47 pp. M. 2.50

PLAUT (H. C.). Ueber gemeinsame Teilbarkeit von n Formen einer Variabeln von n linearen homogenen Differential- oder Differenzenausdrücken. Königsberg, 1914. 8vo. 73 pp.

ROBERT OF CHESTER. See AL-KHOWARIZMI.

ROTH (L.). Ueber die singulären Stellen des Haupttangentenkurvensystems einer Fläche. München, 1914. 8vo. 50 pp.

RUNGE (C.). See ENCYKLOPÄDIE.

STRASZEWICZ (S.). Beiträge zur Theorie der konvexen Punktmengen. Zürich, 1914. 8vo. 57 pp.

TALLADA (F.). Consideraciones acerca del espacio. Con discurso de contestacion por E. Fontseré y Riba. Barcelona (*Mem. Acad.*), 1914. 4to. 20 pp. M. 2.00

TIMERDING (H. E.). Die Analyse des Zufalls. Braunschweig, 1915. 8vo. 9+167 pp. M. 5.00

VOLTERRA (V.). The theory of permutable functions. Princeton, N. J., 1915. 8vo. 68 pp. $1.25

WALTER (J. E.). Nature and cognition of space and time. West Newton, Pa., Johnston and Penney, 1915. 186 pp.

WATSON (G. N.). See WHITTAKER (E. T.).

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WHITTAKER (E. T.) and WATSON (G. N.). A course of modern analysis. 2d edition. Cambridge, University Press, 1915. 560 pp. 18s.

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II. ELEMENTARY MATHEMATICS.

DOBBS (F. W.) and MARSDEN (H. K.). Arithmetic. Part 1. (Bell's mathematical series for schools and colleges.) London, Bell, 1915. 8vo. 18+353+24 pp. 3s.

F. G. M. Manuel d'algèbre d'après les programmes de 1902 et de 1912. (Cours de mathématiques élémentaires.) Tours, A. Mame; et Paris, J. de Gigord, 1915. 8vo. 16+562 pp.

MARSDEN (H. K.). See DOBBS (F. W.).

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III. APPLIED MATHEMATICS.

ARNDT (K.). Handbuch der physikalisch-chemischen Technik für Forscher und Techniker. Stuttgart, 1915. Gr. 8vo. 16+830 pp. M. 28.00

AUERBACH (F.). See HANDBUCH.

BALLIF (L.). See SENSEVER (G.).

BAULE (B.). Theoretische Behandlung der Erscheinungen in verdünnten Gasen. Göttingen, 1914. 8vo. 34 pp. M. 1.50

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BROCKMANN (K.). Die Bewegung der Ionen im Glimmstrom. Berlin, 1915. 8vo. 34 pp. M. 2.00

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